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Steady-state response of an elastically supported infinite beam to a moving load

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Abstract

The steady-state response of a uniform beam placed on an elastic foundation and subjected to a concentrated load moving with a constant speed has been investigated. The foundation is modeled by using one and two parameters. The mathematical form of the solution is justified by Fourier transform. It is observed that the steady state is not attained at supercritical speed of the load in the ideal undamped case. Numerical results are presented for maximum settlement, uplift and bending moment in the beam. The effect of difference in the modeling of the foundation is shown to be insignificant.

1. Introduction

The increased speeds of modern trains are normally accompanied by increased transient movement of rail and ground, which may cause noticeable vibration and large deflection in the rail track along with structure-borne noise in the nearby buildings. For modern high-speed trains these transient movements are especially high when train speeds approach certain critical velocities in the track-ground system. There are two main critical wave velocities in the track ground system: the velocity of Rayleigh surface wave in the ground and the minimum phase velocity of the bending wave propagating in the track supported by the ballast, the latter velocity

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being referred to as the track critical velocity. Modern high-speed trains, especially in the case of very soft soil, can easily exceed both these velocities. Krylov [1] predicted that if the train velocity (v) exceeds the Rayleigh wave velocity in the supporting soil, then a ground vibration boom occurs which is associated with a very large increase in generated ground vibrations, as compared with the case of conventional trains. This phenomenon is similar to sonic boom for aircraft crossing the sound barrier. As a result, the structures over which the trains move are subjected to larger dynamic stresses and wave radiation effects become more and more important for the development of high speed train tracks. It is thus necessary to analyze and understand the behavior of these structures and design them properly against the additional stress. A beam resting on a soil mass can be conveniently used to represent the track of railway and rocket testing facilities, as well as the pavements used as roadways and in airports. Such a continuum model is often used to represent flexible structures that are essentially one dimensional in geometry. The ground may be represented by foundation models like the one parameter Winkler model or the two parameter models developed by researchers like Pasternak [2] and Vlasov and Leotiev [3] or higher order models like those developed by Kerr [4] and Reissner [5]. In the present analysis, Pasternak [2] model has been used to represent the ground.

The differential equation governing the system of moving loads on beams on elastic foundation can be obtained by considering the dynamic equilibrium of beam resting on ground and undergoing transverse vibrations. Depending upon the physical system, either an Euler–Bernoulli beam or a Timoshenko beam may be considered for this purpose. In the present analysis, an Euler–Bernoulli beam has been considered, because the depth and hence the rotary inertia of the track can be considered small as compared to the translational inertia.

The analytical solution of the problem of moving load has been solved by only a few researchers [6,7]. Kenny [7] included the effect of damping and obtained amplification factors to study an infinite beam on Winkler foundation. The beam was idealized using the Euler–Bernoulli beam theory and the analytical solution for the steady response of the beam was obtained. The velocity of propagation of free waves for the undamped case was obtained and it was shown that if the velocity of traveling load is equal to this free wave velocity, then displacement increases without bound resulting in resonance. He also investigated the effect of viscous damping and discussed the limiting case of no damping.

Solution for the same problem was also obtained by Fryba [6]. He analyzed the response of an unbounded elastic body subjected to a moving load using the technique of triple Fourier integral transformation. From the closed form solution distinct differences can be obtained between the responses in the subsonic, transonic and supersonic cases. Fryba [6] presented a detailed solution for the problem of a constant moving load along an infinite beam on an elastic foundation considering all possible speeds and values of viscous damping. Based on the concept of equivalent stiffness of the supporting structures, a critical speed was identified for the moving load, at which the response of the undamped beam becomes infinite. For load speed lower than the critical speed, the largest amplitude of wave occurs near the point of loading. On the other hand, for load speed higher than the critical speed, the waves moving ahead of the load are smaller in amplitude and wavelength than those behind the load.

Sun [8] presented a closed form solution, for the response of a beam resting on a Winkler foundation subjected to a moving line load, by means of two-dimensional Fourier transform and using Green's function. The response of the beam is studied for the damped case at subcritical speeds.

In the present analysis, an infinite Euler–Bernoulli beam of constant cross-section resting on an elastic foundation is considered. The beam and foundation are assumed to be homogeneous and isotropic. The foundation is modeled using both one parameter and two parameters. The beam is subjected to a constant point load moving with a constant speed along the beam. An effort has been made to find the solution of the governing differential equation analytically with and without viscous damping. The equations governing different responses such as beam deflection, bending moment and shear force have been obtained in closed form for the undamped case. For the case where damping is also present, numerical results are obtained for both underdamping and overdamping cases.

2. Modeling of beams on elastic foundation subjected to moving loads

The differential equation of motion for an Euler–Bernoulli beam, resting on a two-parameter foundation and subjected to a moving load is given by:

$$EI\frac{\partial^4 w}{\partial x^4} - k_1\frac{\partial^2 w}{\partial x^2} + kw + \rho\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = P(x,t), \tag{1}$$

where $w = w(x,t)$ is the transverse deflection of the beam (m), E the Young's modulus of beam material (N/m^2), I the second moment of area of the beam cross section about its neutral axis (m^4), k the spring constant (first parameter) of the soil per unit beam length (N/m^2), k_1 the shear parameter (second parameter) of the soil (N), ρ the mass per unit length of the beam (kg/m), c the coefficient of viscous damping per unit length of the beam (N s/m^2), $P(x,t)$ the applied moving load per unit length (N/m), x the space coordinate measured along the length of the beam (m), t the time (s).

If a concentrated load P moves with a constant velocity v, then $P(x,t) = P\delta(x - vt)$ where δ is the Dirac's delta function and x is measured from the location of the load at $t = 0$.

Eq. (1) may be used for a one parameter foundation model by neglecting the term involving the shear parameter k_1. While using the two parameter model, the values of k and k_1 are based on the constrained deformation of an elastic layer given by Vlazov and Leotiev [3]. For a single layer of thickness H with a linear variation of normal stresses, k and k_1 per unit width are given, respectively, by

$$\bar{k} = \frac{E_s}{H(1+v_s)(1-2v_s)}, \qquad \bar{k}_1 = \frac{E_s H}{6(1+v_s)}. \tag{2}$$

The values of E_s (Young's modulus) and v_s (Poisson's ratio) can be determined from triaxial tests.

The assumptions made in deriving Eq. (1) are:

(i) The beam is initially straight.
(ii) The beam and soil materials are linearly elastic with same moduli in tension and compression.
(iii) Structural deformations are small.

(iv) Shear deformation and rotary inertia are negligible.
(v) Inertial forces of the vehicles are much smaller than its dead weight (force of constant magnitude).

3. Infinite beam on a two parameter foundation model

Divide both sides of Eq. (1) by EI to get

$$\frac{\partial^4 w}{\partial x^4} - \frac{k_1}{EI}\frac{\partial^2 w}{\partial x^2} + \frac{k}{EI}w + \frac{\rho}{EI}\frac{\partial^2 w}{\partial t^2} + \frac{c}{EI}\frac{\partial w}{\partial t} = \frac{P}{EI}\delta(x - vt). \tag{3}$$

Now define

$$a = \frac{\rho}{2EI}, \quad b^2 = \frac{k}{EI}, \quad c_1 = \frac{k_1}{2EI} \quad \text{and} \quad d = \frac{c}{EI}$$

when Eq. (3) can be written as

$$\frac{\partial^4 w}{\partial x^4} - 2c_1\frac{\partial^2 w}{\partial x^2} + b^2 w + 2a\frac{\partial^2 w}{\partial t^2} + d\frac{\partial w}{\partial t} = \frac{P}{EI}\delta(x - vt). \tag{4}$$

For an infinite beam in the steady state, it is shown in the appendix that the response w becomes a function of $(x-vt)$, rather than of (x, t). Consequently, Eq. (4) can be put in a homogeneous form and the external load can be treated as a jump in the shear force and can be included in the boundary condition. Thus writing $\xi = x - vt$, Eq. (4) is reduced to the following ordinary differential equation:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}\xi^4} - 2c_1\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} + b^2 w + 2av^2\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} - \mathrm{d}v\frac{\mathrm{d}w}{\mathrm{d}\xi} = 0. \tag{5}$$

Let us assume that the solution of the above equation as

$$w = \mathrm{e}^{m\xi}$$

which after substituting in Eq. (5) yields

$$m^4 - 2c_1 m^2 + b^2 + 2av^2 m^2 - dvm = 0. \tag{6}$$

3.1. Underdamped case

Defining critical damping coefficient $d_{\mathrm{cr}} = 2\sqrt{2}b\sqrt{a}$, $d < d_{\mathrm{cr}}$ constitutes the underdamped case. It can be shown, that with underdamping, the four roots of Eq. (6) are of the form:

$$\begin{aligned} m_1 &= -p + \mathrm{i}q, \\ m_2 &= -p - \mathrm{i}q, \\ m_3 &= p + \mathrm{i}r, \\ m_4 &= p - \mathrm{i}r, \end{aligned} \tag{7}$$

where p, q and r are real positive numbers.

Thus solution of the differential equation (5) is given by

$$w_1(\xi) = \mathrm{e}^{-p\xi}[A \cos q\xi + B \sin q\xi] \quad \text{for } \xi > 0 \text{ to ensure that } w_1 \text{ vanishes as } \xi \to \infty, \tag{8}$$

$$w_2(\xi) = \mathrm{e}^{p\xi}[C \cos r\xi + D \sin r\xi] \quad \text{for } \xi < 0 \text{ to ensure that } w_2 \text{ vanishes as } \xi \to -\infty.$$

The wavelength of the front wave is given by $2\pi/q$ and that for the rear wave is given by $2\pi/r$, with $q > r$. The boundary conditions are

$$\begin{aligned} w_1(0) &= w_2(0), \\ w_1'(0) &= w_2'(0), \\ w_1''(0) &= w_2''(0), \\ w_1'''(0) - w_2'''(0) &= \frac{P}{EI}, \end{aligned} \tag{9}$$

where the prime denotes differentiation with respect to ξ. Using Eqs. (8) and (9) one gets

$$A - C = 0, \tag{10}$$

$$-2pA + Bq - Dr = 0, \tag{11}$$

$$\left(r^2 - q^2\right)A - 2pqB - 2prD = 0, \tag{12}$$

$$p\left(3q^2 - 2p^2 + 3r^2\right)A + q\left(3p^2 - q^2\right)B + r\left(r^2 - 3p^2\right)D = \frac{P}{EI}, \tag{13}$$

to solve for the four constants A, B, C and D. The values of A, B, C and D have been solved analytically in the present study.

3.2. Overdamped case

For the overdamped case, i.e., $d > d_{\mathrm{cr}}$, the roots of Eqs. (6) are of the form:

$$\begin{aligned} m_1 &= -p + \mathrm{i}q, \\ m_2 &= -p - \mathrm{i}q, \\ m_3 &= r, \\ m_4 &= s, \end{aligned} \tag{14}$$

where p, q, r and s are real positive numbers.

In this case the solution of Eq. (5) can be written as

$$\begin{aligned} w_1 &= \mathrm{e}^{-p\xi}[A \cos q\xi + B \sin q\xi] \quad \text{for } \xi > 0, \\ w_2 &= C\mathrm{e}^{r\xi} + D\mathrm{e}^{s\xi} \qquad \text{for } \xi < 0. \end{aligned} \tag{15}$$

The four boundary conditions are the same as given by Eq. (9). Using Eqs. (9) and (15), one gets

$$A - C - D = 0, \tag{16}$$

$$-pA + qB - rC - sD = 0, \tag{17}$$

$$\left(p^2 - q^2\right)A - 2pqB - r^2C - s^2D = 0, \tag{18}$$

$$p\left(3q^2 - p^2\right)A + q\left(3p^2 - q^2\right)B - r^3C - s^3D = \frac{P}{EI}. \tag{19}$$

The values of the constants A, B, C, and D have been solved analytically in the present study.

3.3. Undamped case (velocity less than critical)

We define a critical velocity of the moving load (see Section 3.4), $v_{cr} = \sqrt{(b + c_1)/a}$ and consider the case $v < v_{cr}$. In the absence of damping ($d = 0$), Eq. (6) is reduced to the following bi-quadratic equation:

$$m^4 + \left(2av^2 - 2c_1\right)m^2 + b^2 = 0. \tag{20}$$

The roots of the above equation are

$$m^2 = -\left(av^2 - c_1\right) \pm \sqrt{\left(av^2 - c_1\right)^2 - b^2}, \tag{21}$$

which can also be written as

$$m^2 = -\left(av^2 - c_1\right) \pm \mathrm{i}\sqrt{b^2 - \left(av^2 - c_1\right)^2}, \tag{22}$$

with $av^2 - c_1 < b_1$, i.e., $v < \sqrt{(b + c_1)/a}$ or $v < v_{cr}$.

Now substitute

$$\alpha = \sqrt{\frac{b - (av^2 - c_1)}{2}} \text{ and } \beta = \sqrt{\frac{b + (av^2 - c_1)}{2}}. \tag{23}$$

Thus, the four roots of Eq. (20) will be in the form of complex conjugates and can be written as

$$\begin{aligned} m_1 &= \alpha + \mathrm{i}\beta, \\ m_2 &= \alpha - \mathrm{i}\beta, \\ m_3 &= -\alpha + \mathrm{i}\beta, \\ m_4 &= -\alpha - \mathrm{i}\beta. \end{aligned} \tag{24}$$

It is more suitable to assume the solution in the form of sin and cos functions with exponential term.

Thus, the solution of Eq. (5) with $d = 0$, can be written as

$$\begin{aligned} w_1(\xi) &= \mathrm{e}^{-\alpha\xi}[C_1 \sin \beta\xi + C_2 \cos \beta\xi] + \mathrm{e}^{\alpha\xi}[C_3 \sin \beta\xi + C_4 \cos \beta\xi] \quad \text{for } \xi > 0, \\ w_2(\xi) &= \mathrm{e}^{-\alpha\xi}[D_1 \sin \beta\xi + D_2 \cos \beta\xi] + \mathrm{e}^{\alpha\xi}[D_3 \sin \beta\xi + D_4 \cos \beta\xi] \quad \text{for } \xi < 0. \end{aligned} \tag{25}$$

To ensure zero deflection at infinity the solutions must be in the form (as $\alpha > 0$)

$$w_1(\xi) = e^{-\alpha\xi}[C_1 \sin \beta\xi + C_2 \cos \beta\xi] \quad \text{for } \xi > 0,$$
$$w_2(\xi) = e^{\alpha\xi}[D_3 \sin \beta\xi + D_4 \cos \beta\xi] \quad \text{for } \xi < 0. \tag{26}$$

The boundary conditions are still given by Eqs. (9).

After solving Eqs. (26) with the help of Eqs. (9), the final solution is obtained as

$$w_1(\xi) = \frac{Pe^{-\alpha\xi}}{4EIb\alpha\beta}[\alpha \sin \beta\xi + \beta \cos \beta\xi] \quad \text{for } \xi > 0,$$
$$w_2(\xi) = \frac{Pe^{\alpha\xi}}{4EIb\alpha\beta}[\beta \cos \beta\xi - \alpha \sin \beta\xi] \quad \text{for } \xi < 0. \tag{27}$$

The derivation of Eqs. (27) through transform method is included in Appendix A. The wavelength for both front and rear waves is given by $2\pi/\beta$ and the response is symmetrical about the point of loading.

3.4. Critical velocity

If the velocity of the load $v = \sqrt{(b + c_1)/a}$ then $\alpha = 0$ and consequently both $w_1(\xi)$ and $w_2(\xi)$ shoot up to infinity. This velocity is called the critical velocity, given by

$$v_{\text{cr}} = \sqrt{\frac{b + c_1}{a}}. \tag{28}$$

In terms of original system parameters for the two parameter foundation model

$$v_{\text{cr}2} = \left[\frac{\sqrt{4EIk} + k_1}{\rho}\right]^{1/2}. \tag{29}$$

3.5. Undamped case (velocity greater than critical)

For $v > v_{\text{cr}}$, the roots of Eq. (20) can be written as

$$m^2 = -\left(av^2 - c_1\right) + \sqrt{\left(av^2 - c_1\right)^2 - b^2}$$

and

$$m^2 = -\left(av^2 - c_1\right) - \sqrt{\left(av^2 - c_1\right)^2 - b^2} \tag{30}$$

The four roots of Eq. (30) are as follows:

$$\begin{aligned} m_1 &= iq, \\ m_2 &= -iq, \\ m_3 &= ir, \\ m_4 &= -ir, \end{aligned} \tag{31}$$

where q and r are given by

$$q = \sqrt{\frac{(av^2 - c_1 + b)}{2}} + \sqrt{\frac{(av^2 - c_1 - b)}{2}} \tag{32}$$

and

$$r = \sqrt{\frac{(av^2 - c_1 + b)}{2}} - \sqrt{\frac{(av^2 - c_1 - b)}{2}}, \tag{33}$$

with $q > r$.

The method now runs into essential difficulty, as also noted by Kenny [7]. In the absence of the real part of roots (m's), one cannot choose roots separately for $\xi > 0$ and $\xi < 0$ as was done for the underdamped case. Thus, if one writes

$$w_1(\xi) = C_1 \sin q\xi + C_2 \cos q\xi + C_3 \sin r\xi + C_4 \cos r\xi \quad \text{for } \xi > 0 \tag{34}$$

and

$$w_2(\xi) = D_1 \sin q\xi + D_2 \cos q\xi + D_3 \sin r\xi + D_4 \cos r\xi \quad \text{for } \xi < 0, \tag{35}$$

then eight constants C_1–C_4, D_1–D_4 cannot be determined using only four boundary conditions given by Eq. (9). Furthermore, the deflections at $\xi \to \mp\infty$ do not die down to zero. Therefore, one can conclude that for $v > v_{\mathrm{cr}}$ the steady state (as assumed in the solution procedure) is never attained in perfectly undamped case. In a single degree-of-freedom, undamped system the steady state is never attained when the forcing frequency coincides with the natural frequency. In the present case, the steady state is never attained for all speeds $v > v_{\mathrm{cr}}$.

The response of the undamped beam for $v > v_{\mathrm{cr}}$ can be thought of as the limiting case of the underdamped beam with $d \to 0$. Thus, using Eqs. (8) and (33) one can write

$$w_1(\xi) = A \cos q\xi + B \sin q\xi \quad \text{for } \xi > 0 \tag{36}$$

and

$$w_2(\xi) = C \cos r\xi + D \sin r\xi \quad \text{for } \xi < 0. \tag{37}$$

However, still one gets undiminished harmonic waves both towards left and right violating $w \to 0$ as $\xi \to \mp\infty$.

4. Results and discussion

The data assumed in the present work are given in Table 1. Numerical results are obtained using the analysis presented in Section 3. The term amplification factor is defined as the ratio of maximum value of responses (such as deflection, bending moment and shear force, etc.) in dynamic case to that of the static case. Amplification factors can be written as

$$\text{Deflection Amplification Factor} = \frac{\text{Maximum Deflection (Dynamic)}}{\text{Maximum Deflection (Static)}},$$

$$\text{Bending Moment Amplification Factor} = \frac{\text{Maximum Bending Moment (Dynamic)}}{\text{Maximum Bending Moment (Static)}}.$$

A very important term, the coefficient of characteristic wavelength in the static case can be written as [7]

$$\lambda = \left[\frac{k}{4EI}\right]^{1/4} = \sqrt{b/2} \qquad (m^{-1}).$$

Table 1
Soil and beam parameters

Parameters	Assumed values
ρ (kg/m)	25
EI (N m^2)	1.75×10^6
K (N/m^2)	40.78×10^5
k_1 (N)	666875
P (N/m)	93.36×10^3
E_s (N/m^2)	3.73×10^6
v_S	0.4

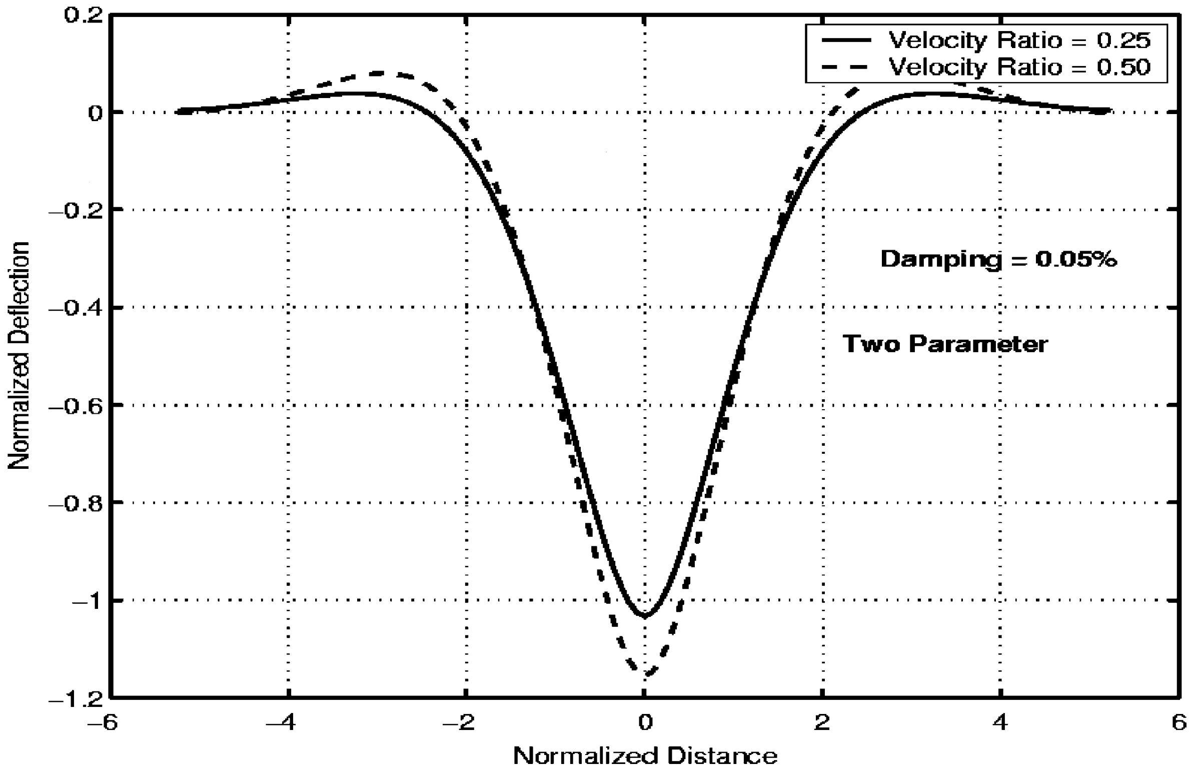
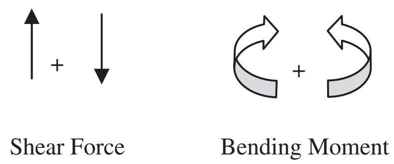


Fig. 1. Normalized deflection vs. normalized distance with damping for velocity ratio = 0.25 and velocity ratio = 0.50, damping = 0.0005 for two parameter model.

The distance along the beam, i.e., the x-axis has been normalized by multiplying the distance (ahead and behind) from the load by λ and the y-axis has been normalized by dividing the responses (deflection, bending moment and shear force) obtained by the maximum value of these in the static case (i.e. at $v = 0$).

The sign convention, which is followed in the present study, can be shown schematically as



The amount of damping is expressed as percentage of the critical damping. The velocity of the load is expressed by the ratio v/v_{cr}.

Figs. 1–7 show the deflection of the beam as a function of the distance from the load for various values of damping, speed of the load (subcritical, critical and supercritical) and two types of foundations. The negative value of the deflection signifies downward deflection with the moving load acting downward signifying settlement whereas the positive deflection implies uplift. The maximum deflection is somewhat more for the two parameter foundation as compared to that for the one parameter foundation. The difference decreases from about 6% to 3% as the velocity

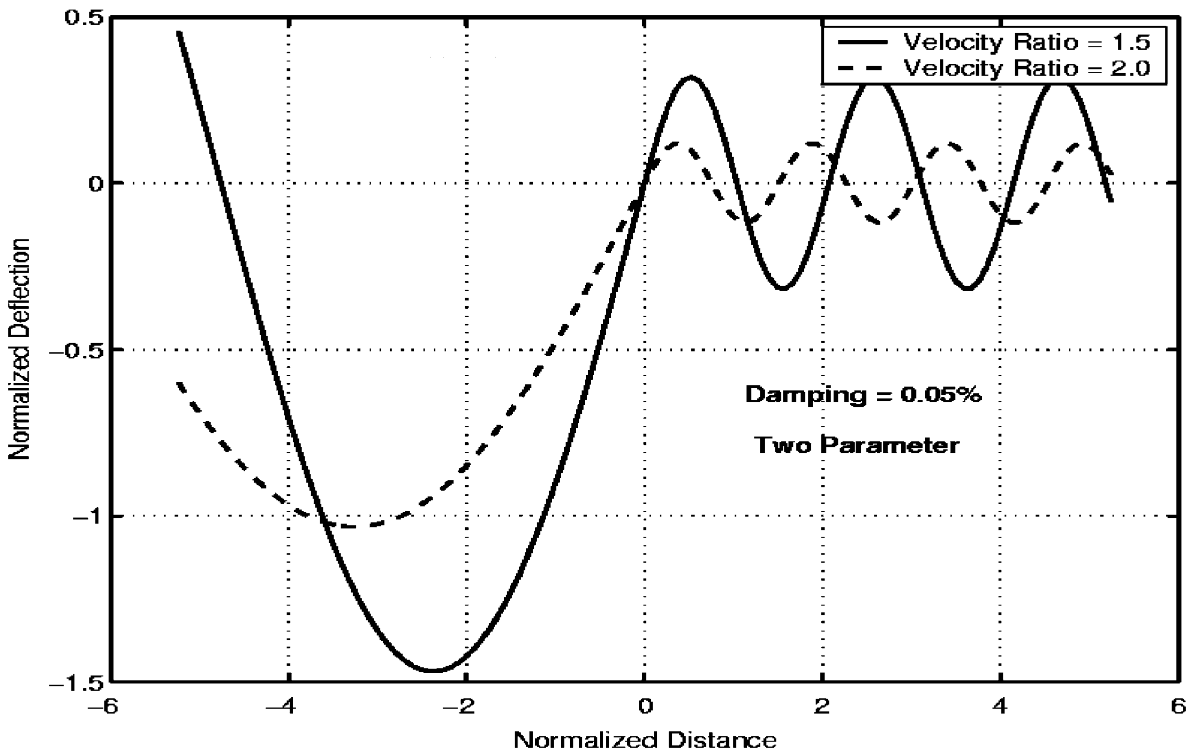


Fig. 2. Normalized deflection vs. normalized distance with damping for velocity ratio = 1.5 and 2.0, damping ratio = 0.0005 for two parameter model.

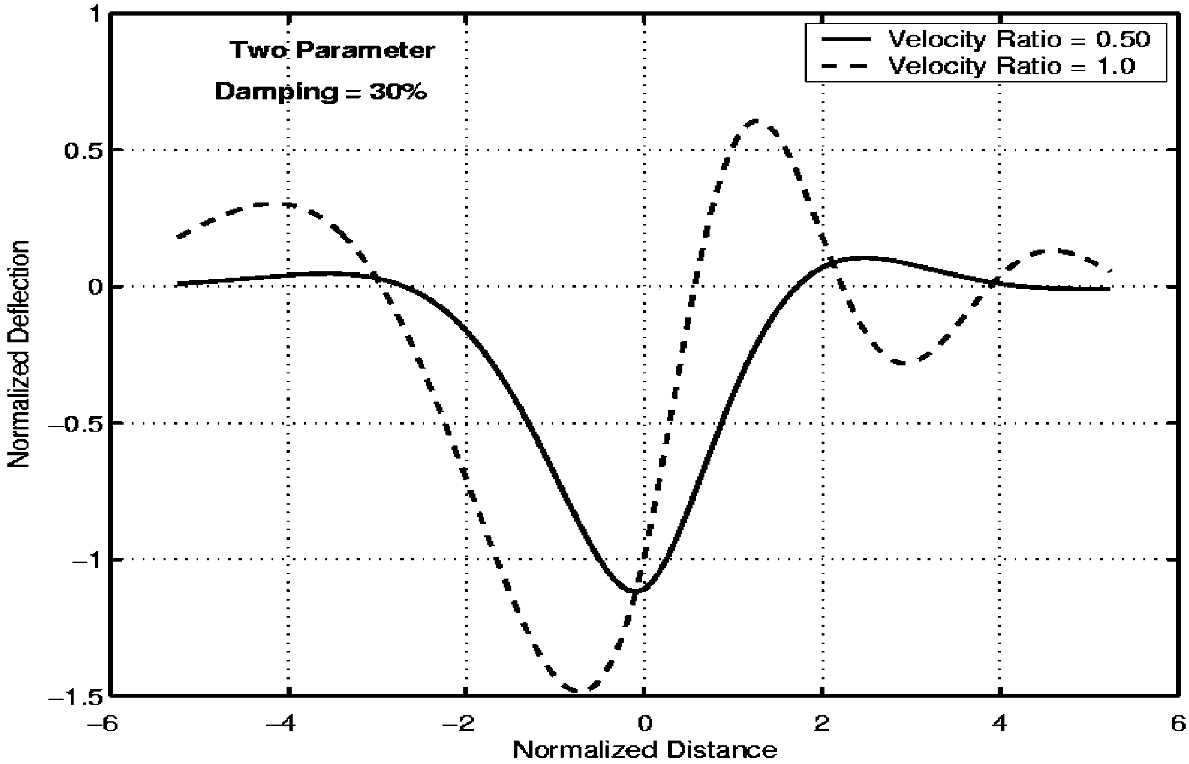


Fig. 3. Normalized deflection vs. normalized distance with damping for velocity ratio = 0.50 and velocity ratio = 1.0, damping = 0.30 for two parameter model.

ratio increases from 1 to 2. With increase in damping, the difference in the response of two types of foundations decreases (see Figs. 4 and 5).

At very low damping, the maximum settlement occurs close to the load and symmetrically dies down on either side of it if $v < v_{cr}$. A small amount of uplift occurs at distances away from the load (Fig. 1). From Fig. 2, it can be concluded that for the supercritical velocity regime, both the maximum settlement and uplift decrease with increasing speed. The point of maximum settlement occurs behind the load and the distance of this point from the load increases with increasing speed. Most of the deflection occurs behind the load as the energy of deformation in the beam foundation system propagates with an average speed (considering all frequencies and wavelengths) lower than that of the load. At a distance away from the load, the deflection (consisting of both settlement and uplift) dies down, the rate of which depends on the amount of damping.

Comparing Figs. 1 and 3 for $v/v_{cr} = 0.5$, one can observe that, as expected, the increased damping decreases the maximum settlement to some extent. However, the maximum uplift is hardly affected. The point of maximum settlement shifts behind the load and the amount of shift increases with increasing speed.

Figs. 4 and 5 show that for the parameter values considered in this paper, depending on the extent of damping, the maximum settlement is about 3.5–6% more for the two parameter

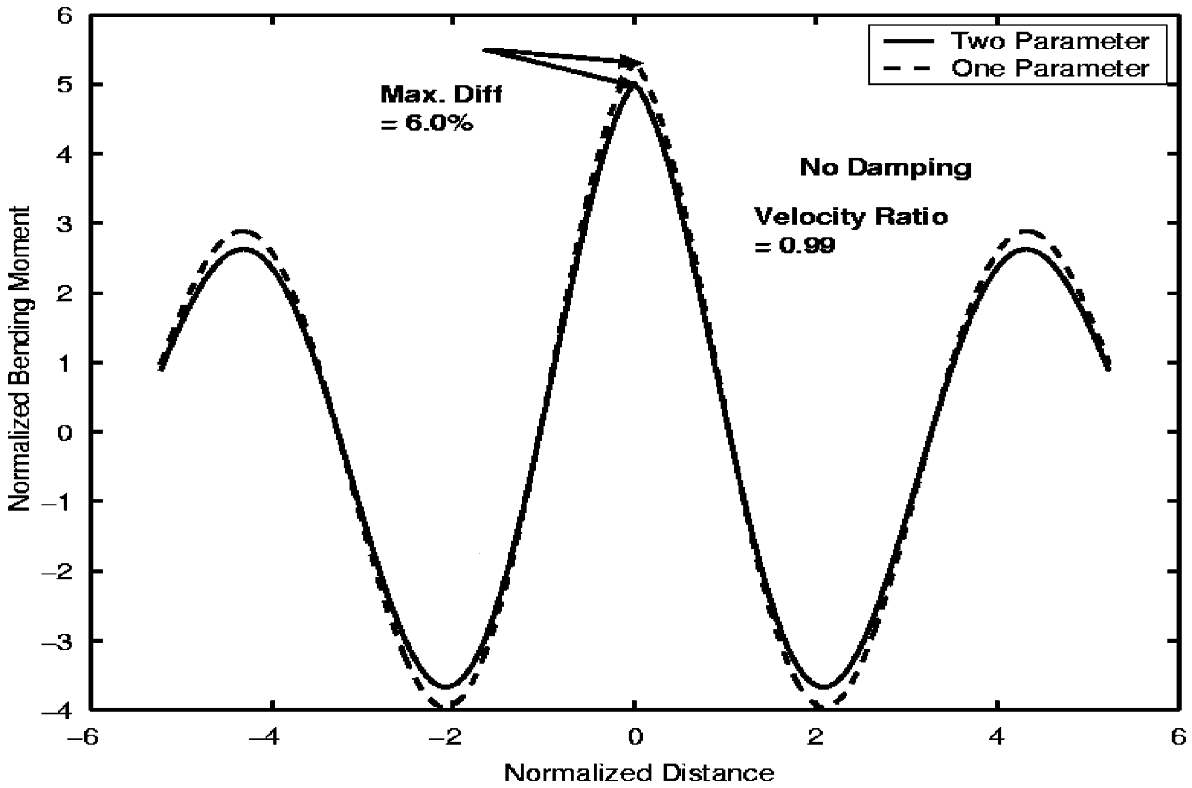


Fig. 4. Normalized bending moment vs. normalized distance without damping for velocity ratio = 0.99 for one and two parameter models.

foundation than that with the one parameter foundation. Numerical results (not shown here) indicated that this amount of increase in the maximum settlement increases with increasing speed. Comparison of Figs. 4 and 5 indicates that with increased damping, the deflection dies down rather rapidly with increasing distance from the loaded point.

The variation of bending moment along the beam is shown in Fig. 7 for subcritical and critical velocities and in Fig. 8 for supercritical velocities. It is clearly seen that the maximum negative bending moment occurs at a point shifted ahead of the load and the amount of shifting decreases with increasing speed. The maximum positive bending moment however occurs near the load so long as $v < v_{cr}$ and shifts to ahead of the load for supercritical velocities. For subcritical velocity, the bending moment distribution is almost symmetrical about the load, whereas for supercritical velocities, the bending moment decreases monotonically behind the load and assumes a high value just ahead of the load and then damps out in an oscillatory fashion.

The variation of dynamic amplification for various quantities such as settlement, uplift, sagging bending moment (ahead, behind and at the load) with increasing speed are shown in Figs. 9–13. Both under and overdamped situations are included. It can be seen that in general, the curves are similar in nature to those of the frequency response curves of a single degree-of-freedom vibratory system with the critical speed being analogous to the resonance frequency.

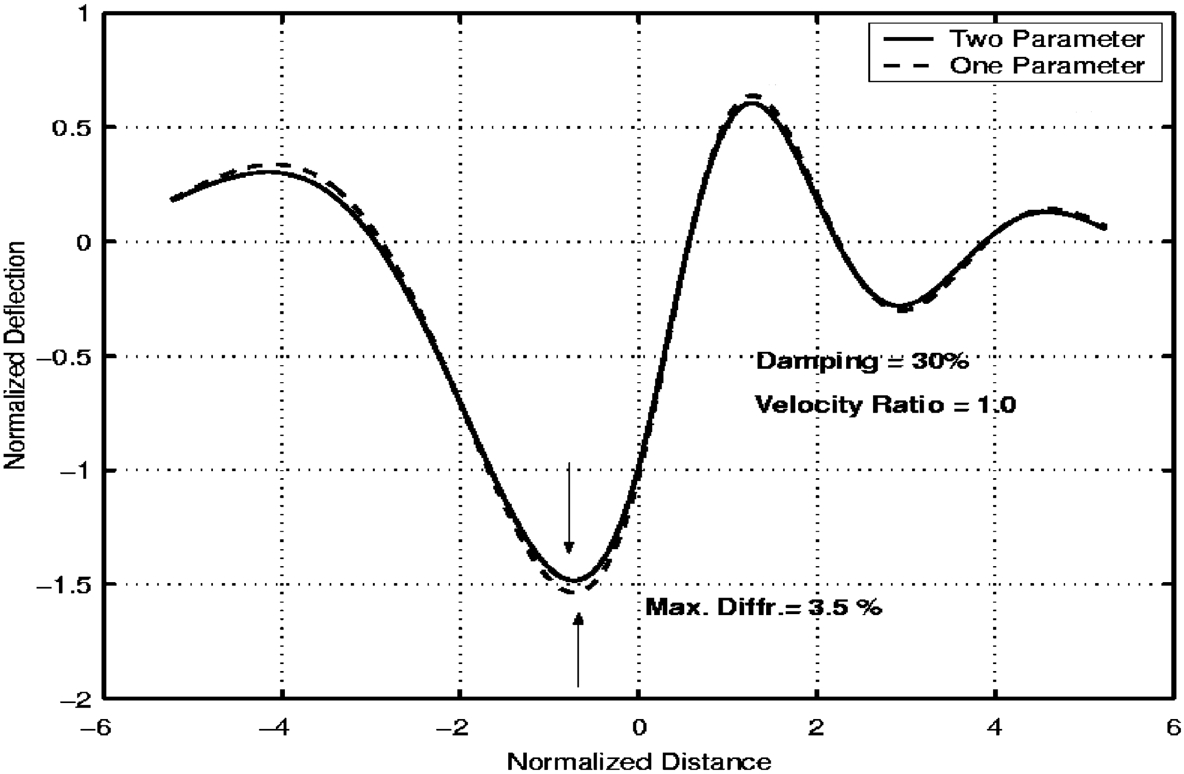


Fig. 5. Normalized deflection vs. normalized distance with damping for velocity ratio = 1.0 damping ratio = 0.30, for one and two parameter models.

5. Conclusions

The major conclusions of the present study are as follows:

1. In the ideal situation of no damping, the steady state in the response is never attained for supercritical speed of the moving load.
2. At supercritical speeds of the load, significant deflection occurs behind the load, but bending moment is higher ahead of the load.
3. The point of maximum deflection shifts behind the load with increasing speed and damping.
4. The variation of dynamic amplification factors (for various parameters like settlement, uplift, bending moment) with speed of moving load resembles the frequency response curve of a single degree-of-freedom vibratory system. The critical speed is analogous to resonance in the latter case. The peaks in both these kinds of plots are controlled in a similar fashion by damping.

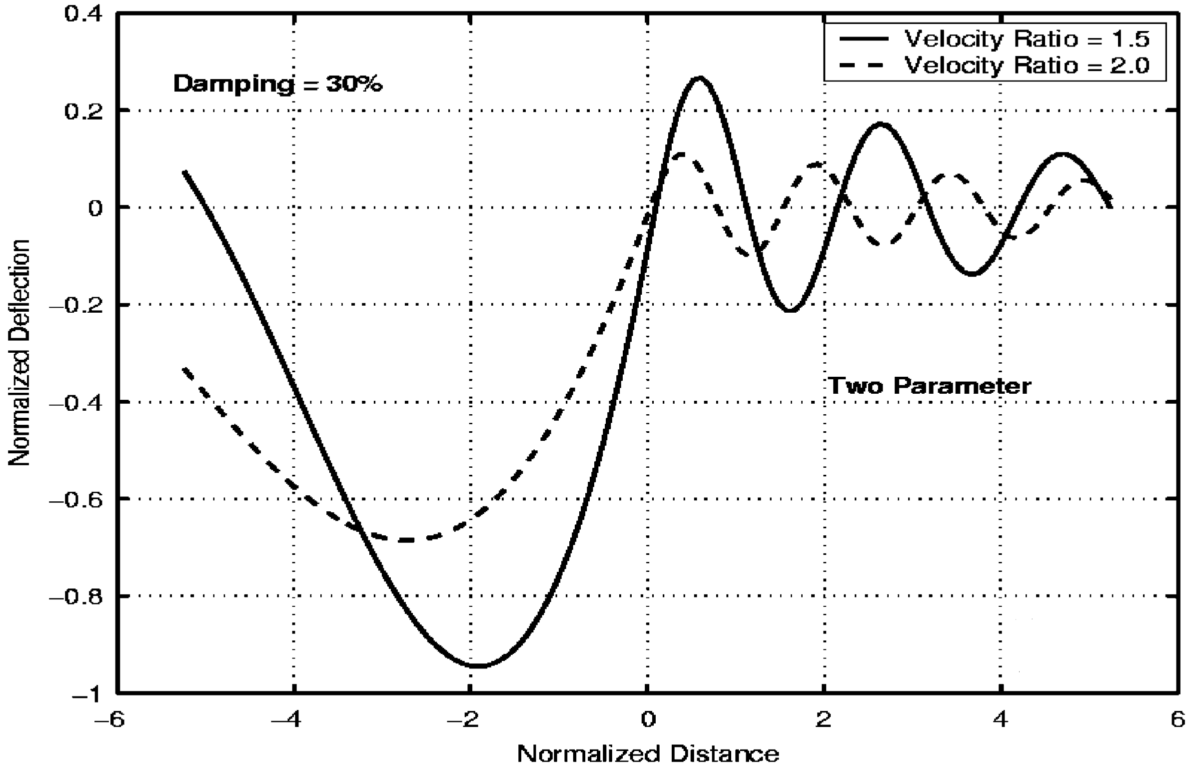


Fig. 6. Normalized deflection vs. normalized distance with damping for velocity ratio = 1.5 and 2.0, damping ratio = 0.30, for two parameter model.

Appendix A

In this appendix the form of the steady-state solution $w(x-vt)$ is justified by using Fourier transform. Towards this end, for simplicity $k_1 = 0$ and $c = 0$ are used, when the equation of motion reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + kw + \rho\frac{\partial^2 w}{\partial t^2} = P\delta(x - vt). \tag{A.1}$$

Define $b^2 = k/EI$, $a = \rho/2EI$ as in Section 3, to get

$$\frac{\partial^4 w}{\partial x^4} + b^2 w + 2a\frac{\partial^2 w}{\partial t^2} = \frac{P}{EI}\delta(x - vt). \tag{A.2}$$

Let

$$w^* = \int_{-\infty}^{\infty} w(x,t)\mathrm{e}^{-\mathrm{i}\gamma x}\,\mathrm{d}x \tag{A.3}$$

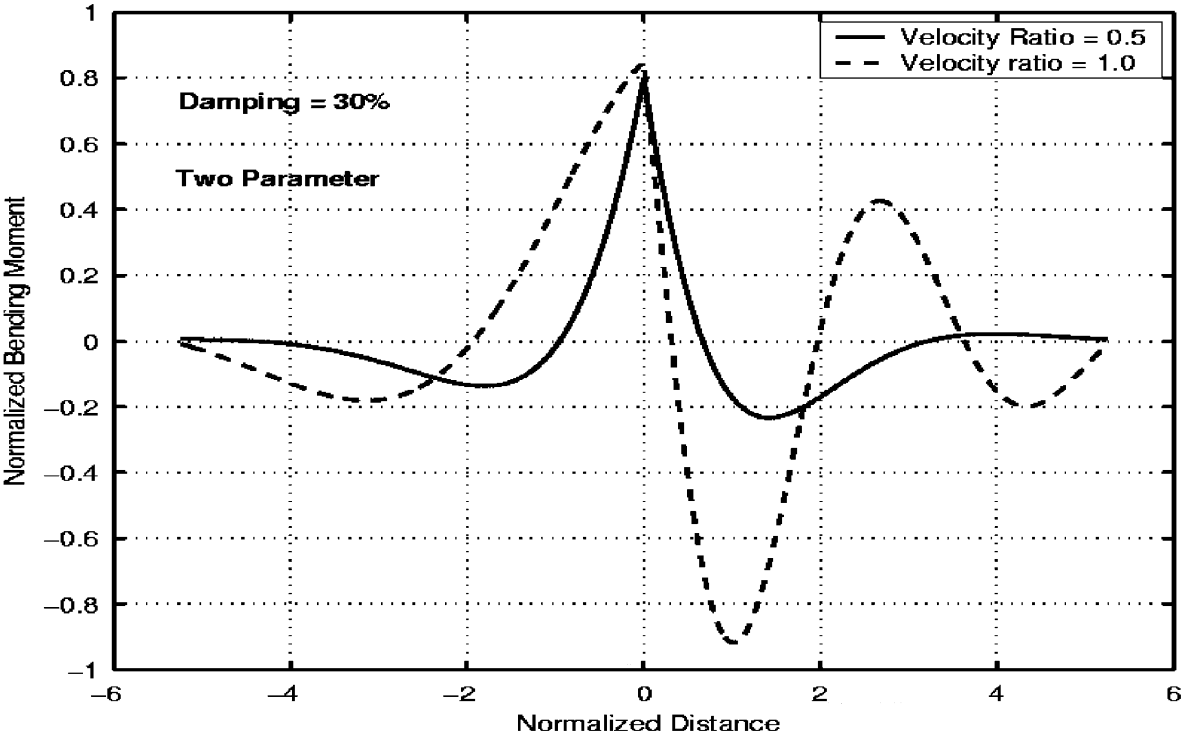


Fig. 7. Normalized bending moment vs. normalized distance with damping for velocity ratio = 0.5 and velocity ratio = 1.0, damping ratio = 0.30 for two parameter model.

and

$$w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w^*(\gamma,t)e^{i\gamma x}\,d\gamma \tag{A.4}$$

constitute a Fourier transform pair. Multiplying both sides of Eq. (A.2) by $e^{-i\gamma x}$ and integrating by parts over x from $-\infty$ to $+\infty$ and assuming w and its space derivatives vanishing at $x = \pm\infty$, one gets

$$\alpha^4 w^* + b^2 w^* + 2a\frac{d^2 w^*}{dt^2} = \frac{P}{EI}e^{-i\gamma vt}. \tag{A.5}$$

The steady-state solution is given by only the particular integral of Eq. (A.5). The complimentary function, i.e., the solution of the homogeneous part (i.e., with right-hand side of Eq. (A.5) equal to zero) dies down in the presence of slightest damping (though neglected here).

Substituting $w^* = W^* e^{-i\gamma vt}$ in Eq. (A.5), one gets

$$(\gamma^4 + b^2 - 2av\gamma^2 v^2)W^* = \frac{P}{EI}, \tag{A.6}$$

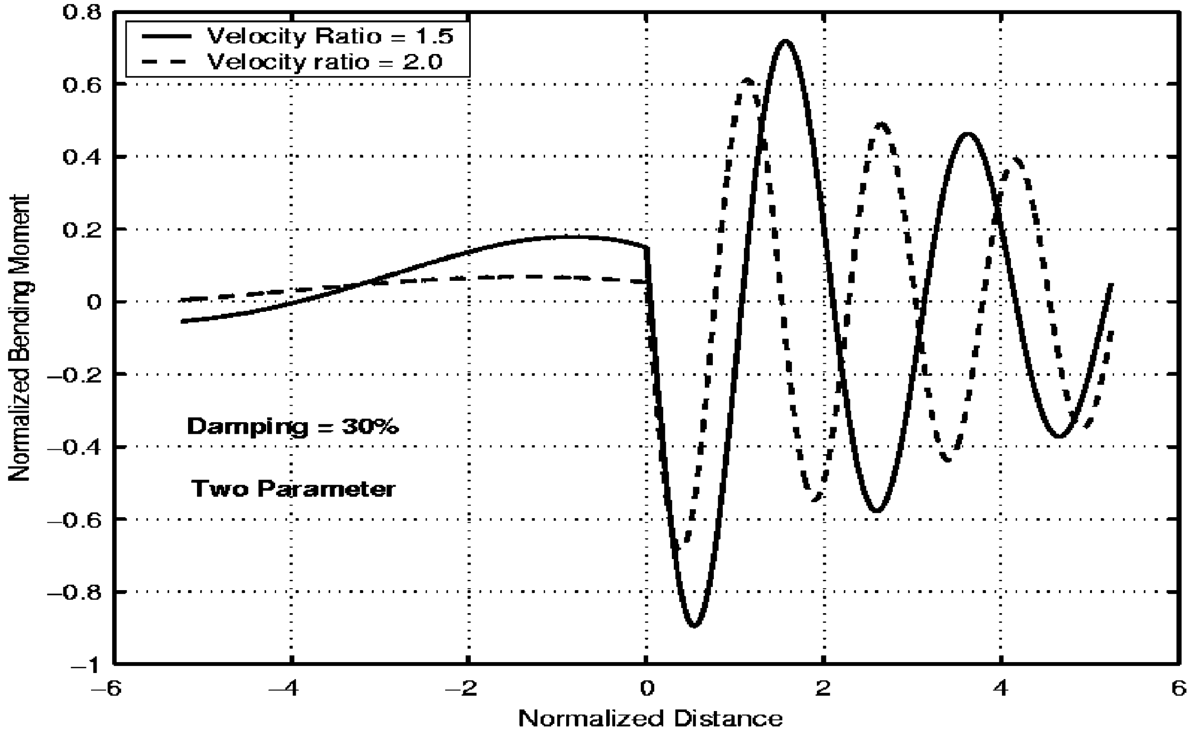


Fig. 8. Normalized bending moment vs. normalized distance with damping for velocity ratio = 0.5 and 1.0, damping ratio = 0.30 for two parameter model.

or

$$W^* = \frac{P}{EI}\left[\frac{1}{\gamma^4 - 2av^2\gamma^2 + b^2}\right], \tag{A.7}$$

or

$$w^* = \frac{P}{EI}\left[\frac{1}{\gamma^4 - 2av^2\gamma^2 + b^2}\right]\mathrm{e}^{-\mathrm{i}\gamma vt}. \tag{A.8}$$

Thus from Eq. (A.4), one gets

$$w(x,t) = \frac{P}{EI}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\gamma(x-vt)}}{\gamma^4 - 2av^2\gamma^2 + b^2}\,\mathrm{d}\gamma. \tag{A.9}$$

The integral in Eq. (A.9) is evaluated by contour integration [9] as discussed below.

$$w(x,t) = (P/EI)(1/2\pi)2\pi\mathrm{i}\sum \text{residues at four simple poles.} \tag{A.10}$$

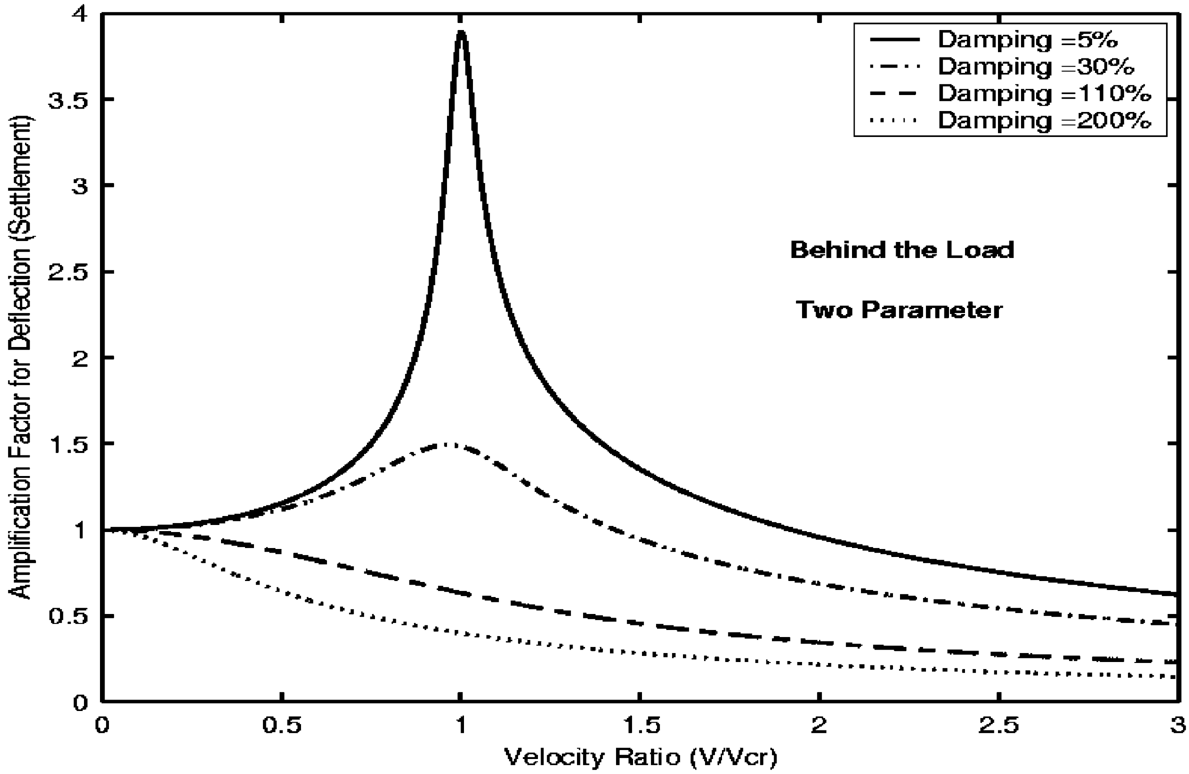


Fig. 9. Amplification factor for deflection (settlement) behind the load for two parameter model.

A.1. Residue calculation

Let

$$q(\gamma) = \gamma^4 - 2av^2\gamma^2 + b^2. \tag{A.11}$$

Poles at γ:

$$q(\gamma) = 0 \Rightarrow \gamma^4 - 2av^2\gamma^2 + b^2 = 0. \tag{A.12}$$

Therefore,

$$\gamma_{1,2}^2 = av^2 \pm \sqrt{(av^2)^2 - b^2}. \tag{A.13}$$

Assume $v^2 < b/a$ where the critical velocity is given by $v_{cr}^2 = b/a$.
For, $v < v_{cr}$

$$\gamma_1^2 = av^2 + \mathrm{i}\sqrt{b^2 - (av^2)^2} \tag{A.14}$$

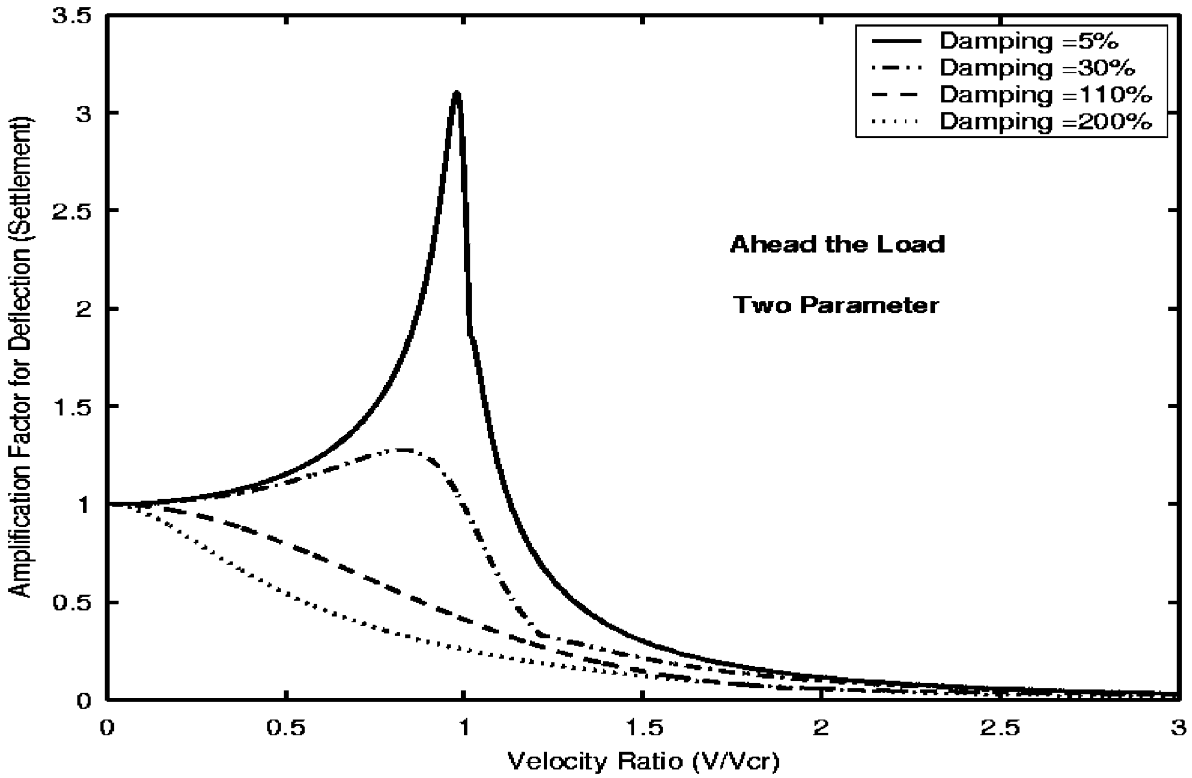


Fig. 10. Amplification factor for deflection (settlement) ahead the load for two parameter model.

and

$$\gamma_2^2 = av^2 - \mathrm{i}\sqrt{b^2 - (av^2)^2}. \tag{A.15}$$

From Eq. (A.14) let the two roots of γ be

$$\gamma_1 = \beta + \mathrm{i}\alpha \tag{A.16}$$

and

$$\gamma_3 = -(\beta + \mathrm{i}\alpha), \tag{A.17}$$

where

$$\alpha^2 + \beta^2 = b, \tag{A.18}$$

$$\beta^2 - \alpha^2 = av^2. \tag{A.19}$$

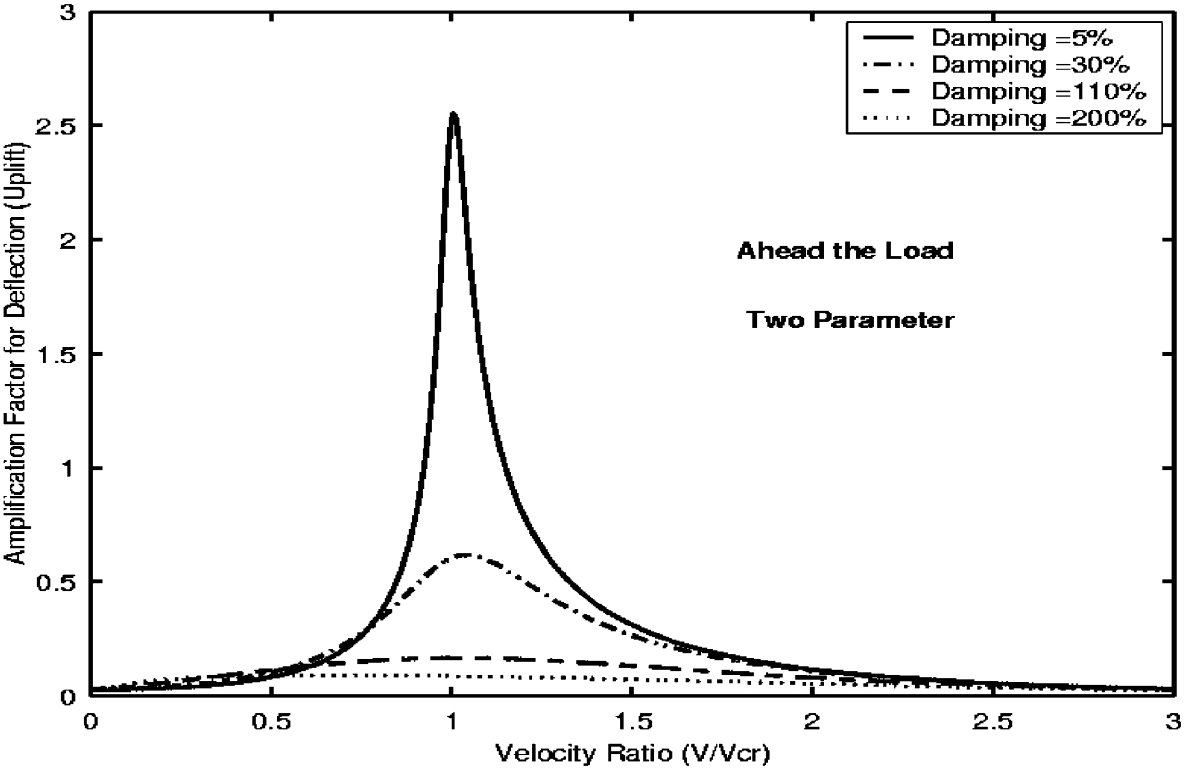


Fig. 11. Amplification factor for deflection (uplift) ahead the load for two parameter model.

Solving Eqs. (A.18) and (A.19)

$$\beta = \sqrt{\frac{b + av^2}{2}}, \qquad \alpha = \sqrt{\frac{b - av^2}{2}}. \tag{A.20}$$

It should be noted that the same values of α and β are obtained from Eqs. (23) in Section 3 with $c_1 = 0$ (as $k_1 = 0$). Similarly the other two roots for γ from Eq. (A.15) are

$$\gamma_2 = -\beta + \mathrm{i}\alpha \tag{A.21}$$

and

$$\gamma_4 = \beta - \mathrm{i}\alpha. \tag{A.22}$$

Thus, the four poles are as shown in Fig. A1, two each on either side of the real axis.

For $\xi(= x - vt) > 0$, one considers the contour C_1 and

$$\begin{aligned} w(x,t) &= \mathrm{i}P/EI\left[\text{Residue at } \gamma_1 + \text{Residue at } \gamma_2\right] \\ &= \frac{\mathrm{i}P}{EI}\left[\frac{\mathrm{e}^{\mathrm{i}\gamma_1\xi}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_1-\gamma_4)} + \frac{\mathrm{e}^{\mathrm{i}\gamma_2\xi}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)(\gamma_2-\gamma_4)}\right]. \end{aligned} \tag{A.23}$$

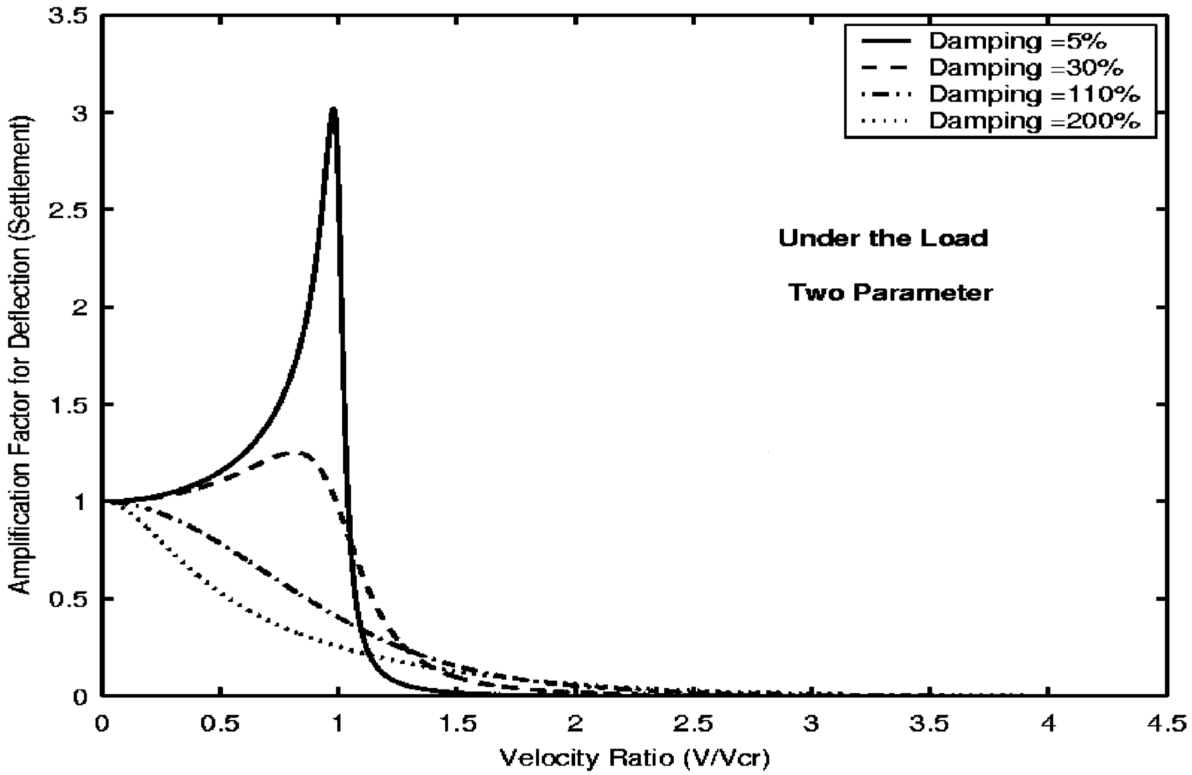


Fig. 12. Amplification factor for deflection (settlement) under the load for two parameter model.

Substituting γ_i's in terms of α and β we finally get

$$w(x,t) = \frac{P\mathrm{e}^{-\alpha\xi}}{4EIb\alpha\beta}[\beta \cos \beta\xi + \alpha \sin \beta\xi]. \tag{A.24}$$

Similarly for $\xi < 0$, considering the contour C_2 (going from $+\infty$ to $-\infty$ on the real axis in order to travel in the counter-clockwise direction along the contour).

$$\begin{aligned} w(x,t) &= \frac{\mathrm{i}P}{EI}\left[\text{Residue at } \gamma_3 + \text{Residue at } \gamma_4\right] \\ &= \frac{P\mathrm{e}^{\alpha\xi}}{4EIb\alpha\beta}[\beta \cos \beta\xi - \alpha \sin \beta\xi]. \end{aligned} \tag{A.25}$$

Thus we note that the steady-state solution is a function of $(x - vt)$ as assumed in Section 3. Moreover, for the undamped beam Eq. (27) is same as Eqs. (A.24) and (A.25) for $\xi > 0$ and $\xi < 0$, respectively. For $v > v_{\mathrm{cr}}$, i.e. $v^2 > b/a$, Eq. (A.13) indicates that the values γ_i's are all real. Thus all the poles lie on the real axis and consequently, the (real) integral corresponding to Eq. (A.9) does not exist [9]. So in the absence of damping for $v > v_{\mathrm{cr}}$, the steady-state response of the form suggested by Eq. (A.6) does not exist. In such a case, there is no justification of neglecting the complimentary function of the solution of Eq. (A.6).

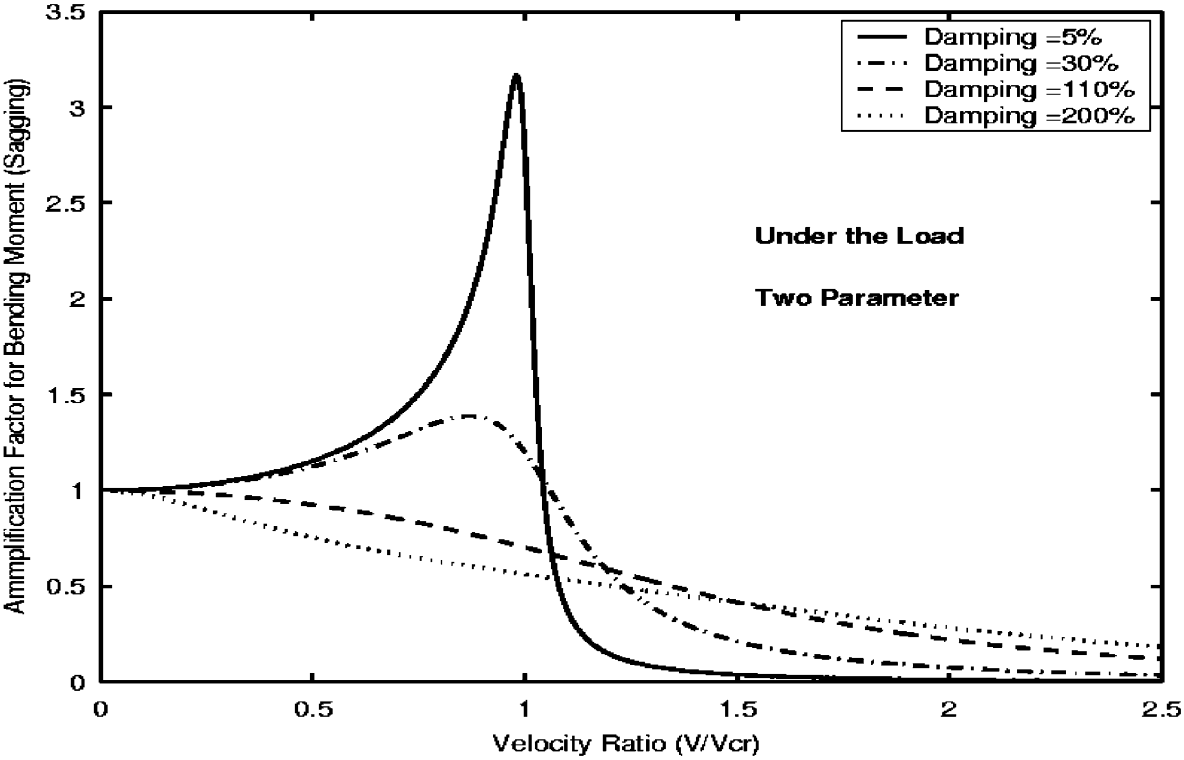


Fig. 13. Amplification factor for bending moment (sagging) under the load for two parameter model.

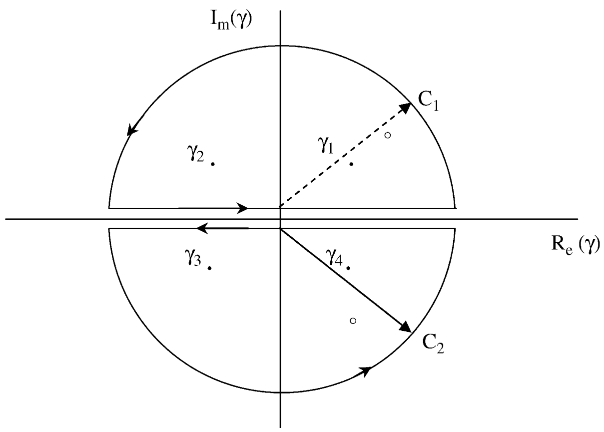


Fig. A1. Position of poles.

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